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A uniformly valid analytic solution of two-dimensional viscous flow over a semi-infinite flat plate

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We apply a new kind of analytic technique, namely the homotopy analysis method (HAM), to give an explicit, totally analytic, uniformly valid solution of the two-dimensional laminar viscous flow over a semi-infinite flat plate governed by $f'''(\eta) + \alpha f(\eta)f''(\eta) + \beta[1 - f'^2(\eta)] = 0$ under the boundary conditions $f(0) = f'(0) = 0$, $f'(+\infty) = 1$. This analytic solution is uniformly valid in the whole region $0 \leqslant \eta < +\infty$. For Blasius' (1908) flow ($\alpha = 1/2$, $\beta = 0$), this solution converges to Howarth's (1938) numerical result and gives a purely analytic value $f''(0) = 0.332057$. For the Falkner–Skan (1931) flow ($\alpha = 1$), it gives the same family of solutions as Hartree's (1937) numerical results and a related analytic formula for $f''(0)$ when $2 \geqslant \beta \geqslant 0$. Also, this analytic solution proves that when $-0.1988 \leqslant \beta < 0$ Hartree's (1937) family of solutions indeed possess the property that $f' \to 1$ exponentially as $\eta \to +\infty$. This verifies the validity of the homotopy analysis method and shows the potential possibility of applying it to some unsolved viscous flow problems in fluid mechanics.

1. Introduction

We consider here the two-dimensional laminar viscous flow over a semi-infinite flat plate, governed by

$$f'''(\eta) + \alpha f(\eta)f''(\eta) + \beta[1 - f'(\eta)f'(\eta)] = 0, \quad \eta \in [0, +\infty), \tag{1.1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{1.2}$$

where $f(\eta)$ is a non-dimensional function related to the stream function $\psi(x, y)$ through

$$f(\eta) = \frac{\psi(x, y)}{(\nu U x)^{1/2}}$$

and the non-dimensional variable η is defined by

$$\eta = y\left(\frac{U}{\nu x}\right)^{1/2}.$$

Here, U is the velocity at infinity, ν is the kinematic viscosity coefficient and x and y are the two independent coordinates.

When $\alpha = 1/2$ and $\beta = 0$, we have the so-called Blasius (1908) equation

$$f'''(\eta) + \tfrac{1}{2} f(\eta) f''(\eta) = 0, \quad \eta \in [0, +\infty), \tag{1.3}$$

under the same boundary conditions (1.2). Weyl (1942*a, b*) proved that there exists a unique solution of the Blasius equation. However, as pointed out by White (1991), although the Blasius equation (1.3) looks rather simple, it seems difficult to find an analytic solution. Blasius (1908) gave the power series solution

$$f(\eta) = \sum_{k=0}^{+\infty} \left(-\tfrac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2} \tag{1.4}$$

where the coefficients A_k are calculated by a recurrence formula

$$A_0 = A_1 = 1, \; A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} \quad (k \geqslant 2). \tag{1.5}$$

However, the value $\sigma = f''(0)$ in (1.4) must be numerically given. Howarth (1938) obtained a numerical result $f''(0) = 0.33206$. So, rigorously speaking, Blasius' solution (1.4) is only a semi-analytic and semi-numerical one. Besides, (1.4) converges in a rather restricted region $|\eta| \leqslant \rho_0$, where $\rho_0 \approx 5.690$. Bairstow (1925) and Goldstein (1930) also gave some series solutions, but the convergence radius of their solutions is finite so that they are invalid for large η. To our knowledge, up to now, no one has reported an explicit, totally analytic, uniformly valid solution of the Blasius equation (1.3). On the other hand, it seems much easier to give its numerical solution. Toepher (1912) and Howarth (1938) applied the Runge–Kutta method to get numerical results. After the appearance of digital computers, Smith (1956), Rosenhead (1963) and Evans (1968) gave very accurate numerical results of the Blasius equation.

When $\alpha = 1$, we get the so-called Falkner–Skan (1931) equation

$$f'''(\eta) + f(\eta) f''(\eta) + \beta[1 - f'(\eta) f'(\eta)] = 0, \quad \eta \in [0, +\infty) \tag{1.6}$$

under the same boundary conditions (1.2). Obviously, (1.6) is more complicated than (1.3) so that it seems more difficult to solve (1.6) analytically. The parameter β can vary in the region $\beta \leqslant 2$ so that Falkner–Skan equation (1.6) describes a class of laminar viscous flows. It is known that in some regions of β, the Falkner–Skan equation has multiple solutions. For large η, Hartree (1937) gave an asymptotic expression for $1 - f'(\eta)$:

$$1 - f'(\eta) \approx A\eta^{-(2\beta+1)} \exp\left(-\tfrac{1}{2}\eta^2\right) + B\eta^{2\beta}, \tag{1.7}$$

where A and B are constants. Noticing that when $\beta \geqslant 0$ the coefficient B must be zero so as to satisfy the boundary condition at infinity and also that the coefficient A can be uniquely determined by the boundary condition at $\eta = 0$, Hartree (1937) pointed out that when $\beta \geqslant 0$ the Falkner–Skan equation (1.6) has a unique solution whose first-order derivative $f'(\eta)$ exponentially tends to 1 as $\eta \to +\infty$. However, when $\beta < 0$, due to (1.7), the boundary condition at infinity is automatically satisfied for any values of A and B so that condition (1.2) does not specify a unique solution. Hartree (1937) found that in the region $\beta_0 < \beta < 0$, where $\beta_0 = -0.19884$ with $f''(\beta_0) = 0$, there exists a family of unique solutions whose first-order derivative tends to 1 exponentially. Hartree's (1937) family of solution has the property $f''(0) \geqslant 0$ and shows neither reversed flow nor velocity overshoot. Further, Stewartson (1954) proved that when $\beta \geqslant 0$ the Falkner–Skan equation (1.6) has indeed a unique solution, and also, when $\beta < \beta_0$ all solutions exhibit the property $f'(\eta^*) > 1$ where $\eta^* > 0$, showing velocity overshoot in some regions. Moreover, in the region $\beta_0 < \beta < 0$, Stewartson (1954) found another new family of numerical solutions exhibiting the property $f''(0) < 0$, which shows reversed flow, whose first-order derivative also seems to tend to 1 exponentially. Unlike Hartree (1937) and Stewartson (1954), Libby & Liu (1967) believed that the overshoot velocity profile might have physical meanings, so they gave other branches of numerical solutions for $\beta < \beta_0$. Their numerical calculations showed that when $\beta < \beta_0$ multiple (probably an infinity) solution to (1.6) and (1.2) exist for any given values of $f''(0)$.

All of above-mentioned solutions to the Falkner–Skan equation (1.6) and (1.2) are given by numerical methods. To our knowledge, no analytic solution has been reported. This unsatisfactory situation might be mainly because our current analytic techniques for nonlinear equations have many limitations and restrictions. In fluid mechanics, the perturbation techniques (see, for example, Nayfeh & Mook 1979) are most widely applied to give analytic approximations of nonlinear problems. For example, as pointed out by Liao (1997*a*), the power series (1.4) can be given by perturbation techniques, too. Another well-known example is Whitehead (1889) and Proudman & Pearson (1957) who applied perturbation techniques to give analytic approximations of the laminar viscous flow past a sphere in a uniform stream. In essence, perturbation techniques use one or more 'small parameters' to transform a nonlinear problem into an infinite sequence of auxiliary linear sub-problems. However, the so-called small parameter greatly restricts application regions of perturbation techniques. First, many nonlinear problems do not contain such a small parameter. Secondly, the validity of perturbation approximations is in general too strongly dependent upon the value of the small parameters. Finally, we have almost no freedom in selecting the related initial guess approximations and governing equations of the related auxiliary sub-problems. Owing to these restrictions, perturbation techniques have failed to solve many important problems in fluid mechanics. For instance, as pointed out by White (1991), neither Whitehead's (1889) straight foreword nor Proudman & Pearson's marching perturbation technique has been successful in giving high-order approximations of the viscous flow past a sphere in a uniform stream, valid in the high Reynolds number region. Therefore, it seems worthwhile developing new kinds of analytic techniques which do not depend upon small parameters at all.

Based on the homotopy method in topology, Liao (1992, 1995, 1997*a*) proposed such a new kind of analytic technique, namely the homotopy analysis method (HAM). The homotopy analysis method has the advantage that in general its validity does not depend upon whether or not nonlinear problems under consideration contain small parameters. Therefore, the homotopy analysis method is valid for more nonlinear problems, especially those with strong nonlinearity. In particular, the homotopy analysis method provides us with great freedom to select initial guess approximations, types of governing equations of auxiliary sub-problems and also some auxiliary parameters. It is just this kind of freedom which makes it easier to ensure that the corresponding approximation sequence given by the homotopy analysis method is convergent. For example, Liao (1997*a*) applied the homotopy analysis method to solve the Blasius equation (1.3) and gave the family of power series solutions in the auxiliary parameter $\hbar$

$$f(\eta) = \lim_{m\to+\infty} \sum_{k=0}^{m} \left[(-\tfrac{1}{2})^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2} \right] \Phi_{m,k}(\hbar), \tag{1.8}$$

$$\eta \in [0, +\infty), \quad -2 < \hbar < 0,$$

where $\sigma = f''(0)$, $A_k (k \geqslant 0)$ is given by (1.5) and the so-called approach function $\Phi_{m,n}(\hbar)$ is defined by

$$\Phi_{m,n}(\hbar) = \begin{cases} 0 & (n > m), \\ (-\hbar)^n \sum\limits_{k=0}^{m-n} \begin{pmatrix} m \\ m-n-k \end{pmatrix} \begin{pmatrix} n+k-1 \\ k \end{pmatrix} \hbar^k & (1 \leqslant n \leqslant m), \\ 1 & (n \leqslant 0), \end{cases} \tag{1.9}$$

As pointed out by Liao (1997*a*), the power series solution (1.8) is valid in the region

$$-\rho_0 < \eta < \rho_0 \left[\frac{2}{|\hbar|} - 1 \right]^{1/3} \quad (-2 < \hbar < 0), \tag{1.10}$$

where $\rho_0 \approx 5.690$ is the convergence radius of the Blasius' power series (1.4). Therefore, different from Blasius' (1908), Bairstow's (1925) and Goldstein's (1930) series solutions, Liao's (1997*a*) power series (1.8) may be valid in the whole region $\eta = [0, +\infty)$ as $|\hbar|$ $(-2 < \hbar < 0)$ tends to zero. Moreover, as pointed out by Liao (1997*a*), the Blasius power series (1.4) is only a special case of (1.8) when $\hbar = -1$, because the approach function $\Phi_{m,n}(\hbar)$ has the interesting property that $\Phi_{m,n}(-1) = 1$ for $n \leqslant m$. Thus, (1.8) is more general than the Blasius' series (1.4). Also, based on the basic ideas of the homotopy analysis method, Liao (1997*b*, *c*, 1998*a*, *b*) also developed some new numerical techniques, such as the so-called 'general boundary element method' and even a non-iterative numerical approach for nonlinear problems. All of these verify the validity and great potential of the homotopy analysis method.

However, rigorously speaking, even Liao's (1997*a*) power series (1.8) is still a semi-analytic and semi-numerical solution, because the value of $f''(0)$ in (1.8) must be obtained by numerical techniques. Notice that Liao (1997*a*) used

$$\mathscr{L}_0 = \frac{\partial^3}{\partial \eta^3} \tag{1.11}$$

as the auxiliary linear operator and $f_0(\eta) = \sigma\eta^2/2$ as the initial guess approximation. The above auxiliary linear operator (1.11) comes directly from the linear term of the Blasius equation (1.3). However, for the homotopy analysis method, this is not necessary, because the homotopy analysis method provides us with a large freedom to select proper auxiliary linear operators and initial guess approximations of many other types. In this paper, we illustrate that, using a more general auxiliary linear operator such as

$$\mathscr{L} = \left(\frac{\partial}{\partial \eta} + \gamma \right) \frac{\partial^2}{\partial \eta^2} = \frac{\partial^3}{\partial \eta^3} + \gamma \frac{\partial^2}{\partial \eta^2}, \tag{1.12}$$

where $\gamma > 0$ is an integer, we can obtain a family of explicit, uniformly valid, totally analytic solutions to (1.1) and (1.2). This is the main purpose of this paper.

This paper has a secondary purpose. Notice that, due to (1.7), when $\beta < 0$, the first-order derivative of the solution to the Falkner–Skan equation (1.6) might tend to 1 either exponentially or algebraically. When $f' \to 1$ exponentially as $\eta \to +\infty$, we have by (1.7) that

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\ln|1-f'(\eta)| = -\eta \tag{1.13}$$

as $\eta \to +\infty$. When $f' \to 1$ algebraically as $\eta \to +\infty$,

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\ln|1-f'(\eta)| = \frac{2\beta}{\eta} \to 0 \tag{1.14}$$

as $\eta \to +\infty$. However, owing to truncation errors of numerical computations, rigorously speaking, it is impossible by means of any numerical methods to exactly examine whether a solution has the property (1.13) for very large η. For example, we analysed numerical solutions to the Falkner–Skan equation (1.6) obtained by Runge–Kutta's method and using the values of $f''(0)$ given by Hartree (1937), Stewartson (1954) and Libby & Liu (1967), and found that as η becomes sufficiently large the term $\mathrm{d}/\mathrm{d}\eta \ln|1-f'(\eta)|$ tends to zero, similar to (1.14) but not to (1.13), for all values of β even including $\beta \geqslant 0$. However, as proved by Hartree (1937) and Stewartson (1954), $f'(\eta) \to 1$ exponentially when $\beta \geqslant 0$. So, numerical methods cannot examine whether or not a numerical solution possesses the property that $f' \to 1$ exponentially as $\eta \to +\infty$. Thus, when $\beta < 0$, none of the numerical methods mentioned above can tell us whether or not $f' \to 1$ exponentially as $\eta \to +\infty$. From (1.7) there might exist an infinite number of possibilities, and $B = 0$ is only one among them. By (1.7), for small but nor-zero $|B|/|A|$, $|1-f'(\eta)|$ would decrease fast at first but then rather slowly as η becomes very large, similar to the situations that all numerical results from the Falkner–Skan equation show. So, we should examine whether or not the solutions to the Falkner–Skan equation (1.6) (when $\beta < 0$) given by the Hartree (1937), Stewartson (1954) and Libby & Liu (1967) indeed have the property that $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$. To do so, we have to analyse its uniformly valid analytic solutions.

2. The explicit, totally analytic solution

In this section, we apply the homotopy analysis method to give an explicit, uniformly valid, totally analytic solution to (1.1) and (1.2). Using $\mathscr{L}$ defined by (1.12) as an auxiliary linear operator, we construct the family of differential equations

$$\begin{aligned}(1-p)\mathscr{L}[F(\eta,p)-f_0(\eta)] \\ = p\hbar\left\{\frac{\partial^3 F(\eta,p)}{\partial\eta^3} + \alpha F(\eta,p)\frac{\partial^2 F(\eta,p)}{\partial\eta^2} + \beta\left[1-\frac{\partial F(\eta,p)}{\partial\eta}\frac{\partial F(\eta,p)}{\partial\eta}\right]\right\}, \\ \eta \in [0,+\infty), \quad \hbar \neq 0, \quad p \in [0,1],\end{aligned} \tag{2.1}$$

with boundary conditions

$$F(0,p) = F'(0,p) = 0, \quad F'(+\infty,p) = 1, \quad p \in [0,1], \tag{2.2}$$

where the prime denotes the partial derivative with respect to η, and

$$f_0(\eta) = \eta - \frac{[1-\exp(-\gamma\eta)]}{\gamma} + \frac{\varpi[1-(1+\gamma\eta)\exp(-\gamma\eta)]}{\gamma^2} \tag{2.3}$$

is an initial guess approximation which satisfies the boundary conditions (1.2), $\hbar$ an auxiliary parameter, and p an embedding parameter. Clearly, at $p = 0$, we have by

(2.1) and (2.2) that

$$F(\eta, 0) = f_0(\eta), \quad \eta \in [0, +\infty). \tag{2.4}$$

When $p = 1$, (2.1) and (2.2) are the same as (1.1) and (1.2), respectively, so that we have

$$F(\eta, 1) = f(\eta), \quad \eta \in [0, +\infty). \tag{2.5}$$

Therefore, according to (2.4) and (2.5), the variation of p from 0 to 1 is just the continuous variation of $F(\eta, p)$ from the initial guess approximation $f_0(\eta)$ to the unknown solution $f(\eta)$ of (1.1) and (1.2). In topology, this kind of continuous variation is called deformation; $f_0(\eta)$ and $f(\eta)$ are called homotopic. Owing to this, we call (2.1) and (2.2) the zeroth-order deformation equations. Notice that owing to (1.12) the auxiliary linear operator $\mathscr{L}$ is completely determined by the value of γ. And from (2.3) we have $f_0''(0) = \gamma + \varpi$ so that for any fixed value of γ we can change the initial guess approximation $f_0(\eta)$ by selecting different values of ϖ.

Assume that the deformation $F(\eta, p)$ governed by (2.1) and (2.2) is smooth enough so that

$$f_0^{[k]}(\eta) = \left. \frac{\partial^k F(\eta, p)}{\partial p^k} \right|_{p=0} \quad (k \geqslant 1), \tag{2.6}$$

namely the kth-order deformation derivative exists. Then, according to (2.4) and the Taylor formula, we have

$$F(\eta, p) = f_0(\eta) + \sum_{k=1}^{+\infty} \left[\frac{f_0^{[k]}(\eta)}{k!} \right] p^k = \sum_{k=0}^{+\infty} \phi_k(\eta) p^k, \tag{2.7}$$

where we define

$$\phi_0(\eta) = f_0(\eta), \quad \phi_k(\eta) = \frac{f_0^{[k]}(\eta)}{k!} \quad (k \geqslant 1). \tag{2.8}$$

Clearly, the convergence region of the above Maclaurin series depends upon the auxiliary parameter $\hbar$ ($\hbar \neq 0$), the auxiliary linear operator $\mathscr{L}$ which is determined by γ ($\gamma > 0$), and the initial guess approximation $f_0(\eta)$ which is determined by both γ and ϖ. Assume that $\hbar$, ϖ and γ are selected such that the series (2.7) is convergent at $p = 1$. Then, owing to (2.5) and (2.7), we get at $p = 1$ the important relationship

$$f(\eta) = f_0(\eta) + \sum_{k=1}^{+\infty} \frac{f_0^{[k]}(\eta)}{k!} = \sum_{k=0}^{+\infty} \phi_k(\eta) \tag{2.9}$$

between the initial guess approximation $\phi_0(\eta) = f_0(\eta)$ and the unknown solution $f(\eta)$.

There are two ways to get the governing equations of the unknown function $\phi_m(\eta) (m \geqslant 1)$. First, differentiating the zeroth-order deformation equations (2.1) and (2.2) m times with respect to p and then setting $p = 0$ and finally dividing it by $m!$, we have for $m \geqslant 1$ that

$$\mathscr{L}(\phi_m - \chi_m \phi_{m-1}) = G_m(\eta), \quad \eta \in [0, +\infty), \tag{2.10}$$

with boundary conditions

$$\phi_m(0) - \chi_m \phi_{m-1}(0) = \phi_m'(0) - \chi_m \phi_{m-1}'(0) = \phi_m'(+\infty) - \chi_m \phi_{m-1}'(+\infty) = 0, \tag{2.11}$$

where the prime denotes the derivative with respect to η, and

$$\chi_m = \begin{cases} 0 & m = 1 \\ 1, & m > 1, \end{cases} \tag{2.12}$$

$$G_1(\eta) = \hbar\,\{\phi_0'''(\eta) + \alpha\phi_0(\eta)\phi_0''(\eta) + \beta[1 - \phi_0'(\eta)\phi_0'(\eta)]\}, \tag{2.13}$$

$$G_m(\eta) = \hbar\left\{\frac{\mathrm{d}^3\phi_{m-1}(\eta)}{\mathrm{d}\eta^3} + \alpha\sum_{k=0}^{m-1}\phi_k(\eta)\frac{\mathrm{d}^2\phi_{m-1-k}(\eta)}{\mathrm{d}\eta^2} - \beta\sum_{k=0}^{m-1}\frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}\phi_{m-1-k}(\eta)}{\mathrm{d}\eta}\right\}, \quad (m \geqslant 2). \tag{2.14}$$

We call equations (2.10) and (2.11) the mth-order deformation equations ($m \geqslant 1$). The second way to get the governing equations is to substitute the series (2.7), say

$$F(\eta, p) = f_0(\eta) + \sum_{k=1}^{+\infty}\phi_k(\eta)p^k,$$

into the zeroth-order deformation equations (2.1) and (2.2), and then, by equating the terms of the same power of p, we can get the same mth-order deformation equations as (2.10) and (2.11). In general, when nonlinear problems under consideration do not contain transcendental functions of unknowns, the above two ways give the same high-order deformation equations. Otherwise, the second approach either does not work or is less efficient. Thus, the first approach is more general than the second.

Notice that from (2.3) and (2.13) we can first calculate the term $G_1(\eta)$ and then obtain $\phi_1(\eta)$ by solving the corresponding linear first-order deformation equations (2.10) and (2.11) when $m = 1$. Further, we can calculate the term $G_2(\eta)$ from (2.14) and then get $\phi_2(\eta)$ by solving the corresponding linear second-order deformation equations (2.10) and (2.11) when $m = 2$, and so on. In this way, the linear mth-order ($m \geqslant 1$) deformation equations (2.10) and (2.11) can be solved one after the other in order. We emphasize that all of these mth-order ($m \geqslant 1$) deformation equations (2.10) and (2.11) are linear. Therefore, in essence, we transfer the original nonlinear equations (1.1) and (1.2) into an infinite number of linear equations (2.10) and (2.11). However, different from perturbation techniques, this kind of transformation has nothing to do with 'small parameters'. Secondly, all of these mth-order ($m \geqslant 1$) deformation equations possess the same linear operator $\mathscr{L}$ defined by (1.12). Thus, the current approach is rather suitable for symbolic calculations. We apply the symbolic computation software *MATHEMATICA* (Abell & Braselton 1994) to solve the first few deformation equations (2.10) and (2.11) and find, to some surprise, that $\phi_m(\eta)$ can be explicitly expressed in the general form

$$\phi_m(\eta) = \sum_{k=0}^{m+1}\Psi_{m,k}(\eta)\exp(-k\gamma\eta), \quad m \geqslant 0, \tag{2.15}$$

where the $\Psi_{m,k}(\eta)$ are defined by

$$\Psi_{0,0}(\eta) = b_{0,0}^0 + b_{0,0}^1\eta, \tag{2.16}$$

$$\Psi_{0,1}(\eta) = b_{0,1}^0 + b_{0,1}^1\eta, \tag{2.17}$$

$$\Psi_{m,0}(\eta) = b_{m,0}^0, \qquad m \geqslant 1, \tag{2.18}$$

$$\Psi_{m,k}(\eta) = \sum_{i=0}^{2(m+1)-k} b_{m,k}^i\eta^i, \quad m \geqslant 1, \quad 1 \leqslant k \leqslant m+1. \tag{2.19}$$

Knowing the structure (2.15) of $\phi_m(\eta)$, we can rigorously deduce the following recurrence formulae for the coefficients $b_{m,n}^k$ of $\phi_m(\eta)$, where $m \geqslant 1$, $0 \leqslant n \leqslant m+1$ and $0 \leqslant k \leqslant 2(m+1)-n$, as follows:

$$b_{m,0}^0 = \chi_m b_{m-1,0}^0 - \gamma^{-1} \sum_{q=0}^{2m} \Gamma_{m,1}^q \, \mu_{1,1}^q - \sum_{n=2}^{m+1} \left[(n-1)\Gamma_{m,n}^0 \, \mu_{n,0}^0 + \sum_{q=1}^{2(m+1)-n} \Gamma_{m,n}^q (n\mu_{n,0}^q - \mu_{n,0}^q - \gamma^{-1}\mu_{n,1}^q) \right], \tag{2.20}$$

$$b_{m,0}^1 = 0, \tag{2.21}$$

$$b_{m,1}^0 = \chi_m b_{m-1,1}^0 + \gamma^{-1} \sum_{q=0}^{2m} \Gamma_{m,1}^q \, \mu_{1,1}^q + \sum_{n=2}^{m+1} \left[n\Gamma_{m,n}^0 \, \mu_{n,0}^0 + \sum_{q=1}^{2(m+1)-n} \Gamma_{m,n}^q (n\mu_{n,0}^q - \gamma^{-1} \mu_{n,1}^q) \right], \tag{2.22}$$

$$b_{m,1}^k = \chi_m b_{m-1,1}^k + \sum_{q=k-1}^{2m} \Gamma_{m,1}^q \, \mu_{1,k}^q, \quad 1 \leqslant k \leqslant 2m-1, \tag{2.23}$$

$$b_{m,1}^k = \sum_{q=k-1}^{2m} \Gamma_{m,1}^q \, \mu_{1,k}^q, \quad 2m \leqslant k \leqslant 2m+1, \tag{2.24}$$

$$b_{m,n}^k = \chi_m b_{m-1,n}^k - \sum_{q=k}^{2(m+1)-n} \Gamma_{m,n}^q \, \mu_{n,k}^q, \quad 0 \leqslant k \leqslant 2m-n, \quad 2 \leqslant n \leqslant m, \tag{2.25}$$

$$b_{m,n}^k = -\sum_{q=k}^{2(m+1)-n} \Gamma_{m,n}^q \, \mu_{n,k}^q, \quad 2m-n+1 \leqslant k \leqslant 2m-n+2, \quad 2 \leqslant n \leqslant m, \tag{2.26}$$

$$b_{m,m+1}^k = -\sum_{q=k}^{m+1} \Gamma_{m,m+1}^q \, \mu_{m+1,k}^q, \quad 0 \leqslant k \leqslant m+1, \tag{2.27}$$

where

$$\mu_{1,k}^q = \frac{q!}{k!} \frac{(q-k+2)}{\gamma^{q-k+3}}, \quad 0 \leqslant k \leqslant q+1, \quad q \geqslant 0 \tag{2.28}$$

$$\mu_{n,k}^q = \frac{q!}{k!} \frac{1}{(n-1)^{q-k+1}\gamma^{q-k+3}} \left\{ 1 - \left(1 - \frac{1}{n}\right)^{q-k+1} \left[(q-k+2) - (q-k+1)\left(1 - \frac{1}{n}\right) \right] \right\},$$
$$0 \leqslant k \leqslant q, \quad n \geqslant 2, \quad q \geqslant 0 \tag{2.29}$$

and the related coefficient $\Gamma_{m,n}^q$ is defined by

$$\Gamma_{m,1}^q = \hbar(d_{m-1,1}^q + \delta_{m,1}^q + \Delta_{m,1}^q), \quad 0 \leqslant q \leqslant 2m-1, \tag{2.30}$$

$$\Gamma_{m,1}^{2m} = \hbar(\delta_{m,1}^{2m} + \Delta_{m,1}^{2m}), \tag{2.31}$$

$$\Gamma_{m,m+1}^q = \hbar(\delta_{m,m+1}^q + \Delta_{m,m+1}^q), \quad 0 \leqslant q \leqslant m+1, \tag{2.32}$$

and for $2 \leqslant n \leqslant m$,

$$\Gamma^q_{m,n} = \begin{cases} \hbar(d^q_{m-1,n} + \delta^q_{m,n} + \Delta^q_{m,n}), & 0 \leqslant q \leqslant 2m-n, \\ \hbar(\delta^q_{m,n} + \Delta^q_{m,n}), & 2m-n+1 \leqslant q \leqslant 2m-n+2, \\ 0, & \text{otherwise}. \end{cases} \tag{2.33}$$

Here the coefficients $\delta^q_{m,n}$ and $\Delta^q_{m,n}$, where $m \geqslant 1$, $0 \leqslant n \leqslant m+1$, $0 \leqslant q \leqslant 2(m+1)-n$, are defined by

$$\delta^q_{m,n} = \alpha \sum_{k=0}^{m-1} \sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} c^i_{k,j} b^{q-i}_{m-1-k,n-j} \lambda^{q-i}_{m-1-k,n-j}, \tag{2.34}$$

and

$$\Delta^q_{m,n} = -\beta \sum_{k=0}^{m-1} \sum_{j=\max\{0,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} a^i_{k,j}\, a^{q-i}_{m-1-k,n-j}, \tag{2.35}$$

respectively, where

$$\lambda^k_{i,j} = \begin{cases} 0, & i = j = 0, \quad k \geqslant 2, \\ 0, & i > 0, \quad j = 0, \quad k \geqslant 1, \\ 0, & j > i+1, \\ 0, & k > 2(i+1)-j, \\ 1, & \text{otherwise}, \end{cases} \tag{2.36}$$

and the related coefficients $a^i_{m,k}$, $c^i_{m,k}$, $d^j_{m,k}$ are given by

$$a^i_{m,k} = (i+1)\lambda^{i+1}_{m,k}\, b^{i+1}_{m,k} - (k\gamma)\, b^i_{m,k}\, \lambda^i_{m,k}, \tag{2.37}$$

$$c^i_{m,k} = (i+1)(i+2) b^{i+2}_{m,k}\, \lambda^{i+2}_{m,k} - 2(k\gamma)(i+1)\, b^{i+1}_{m,k}\, \lambda^{i+1}_{m,k} + (k\gamma)^2\, b^i_{m,k}\, \lambda^i_{m,k}, \tag{2.38}$$

and

$$d^i_{m,k} = (i+1)\lambda^{i+1}_{m,k}\, c^{i+1}_{m,k} - (k\gamma)\, \lambda^i_{m,k}\, c^i_{m,k}. \tag{2.39}$$

The detailed derivation is given in the Appendix.

Using above recurrence formulae, we can calculate all coefficients $b^k_{m,n}$ by using only the first four:

$$b^0_{0,0} = \frac{\varpi}{\gamma^2} - \frac{1}{\gamma}, \quad b^1_{0,0} = 1, \quad b^0_{0,1} = -\frac{\varpi}{\gamma^2} + \frac{1}{\gamma}, \quad b^1_{0,1} = -\frac{\varpi}{\gamma}, \tag{2.40}$$

given by the initial guess approximation (2.3). The corresponding Mth-order approximation of (1.1) and (1.2) is then given by

$$\begin{aligned} f_0(\eta) + \sum_{n=1}^{M} \phi_n(\eta) &= \sum_{m=0}^{M} \sum_{n=0}^{m+1} \Psi_{m,n}(\eta) \exp(-n\gamma\eta) \\ &= t + \left(\sum_{m=0}^{M} b^0_{m,0} \right) + \sum_{n=1}^{M+1} \exp(-n\gamma\eta) \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m+1)-n} b^k_{m,n}\, \eta^k \right). \end{aligned} \tag{2.41}$$

Therefore, we obtain in fact the following *explicit, totally analytic* solution of the two-dimensional laminar viscous flow over a semi-infinite flat plane:

$$f(\eta) = \sum_{k=0}^{+\infty} \phi_k(\eta) = t + \lim_{M\to+\infty}\left[\left(\sum_{m=0}^{M} b_{m,0}^{0}\right) + \sum_{n=1}^{M+1} \exp(-n\gamma\eta)\left(\sum_{m=n-1}^{M}\sum_{k=0}^{2(m+1)-n} b_{m,n}^{k}\eta^k\right)\right]. \quad (2.42)$$

3. The convergence of the analytic solution

In this section, we consider the convergence of the infinite series (2.42). First, we can prove that, if the infinite series

$$\sum_{k=0}^{+\infty} \phi_k(\eta) \quad (3.1)$$

is convergent, it must converge to the solution of equations (1.1) and (1.2).

To prove this, we first assume that $\hbar$, ϖ and γ are selected such that the series (3.1) converges. Then, it must hold that

$$\lim_{m\to+\infty} \phi_m(\eta) = 0. \quad (3.2)$$

Further, we have by (2.10) and (2.12) that

$$\sum_{k=1}^{m} G_m(\eta) = \sum_{k=1}^{m} \mathscr{L}[\phi_k - \chi_k\phi_{k-1}] = \mathscr{L}\left[\sum_{k=1}^{m}\phi_k - \sum_{k=1}^{m}\chi_k\phi_{k-1}\right] = \mathscr{L}\phi_m.$$

Thus, owing to (1.12), (3.2) and above expression, we have

$$\lim_{m\to+\infty}\sum_{k=1}^{m} G_k(\eta) = \lim_{m\to+\infty}\mathscr{L}[\phi_m(\eta)] = 0, \quad \eta\in[0,+\infty), \quad (3.3)$$

say, the infinite sequence $s_1, s_2, s_3, \ldots$, where $s_m = \sum_{k=1}^{m} G_k(\eta)$, converges to zero. On the other hand, owing to (2.13) and (2.14), we have

$$\sum_{i=1}^{m} G_i(\eta) = \hbar\sum_{i=1}^{m}\left\{\frac{d^3\phi_{i-1}(\eta)}{d\eta^3} + \alpha\sum_{k=0}^{i-1}\phi_k(\eta)\frac{d^2\phi_{i-1-k}(\eta)}{d\eta^2} + \beta\left[(1-\chi_i) - \sum_{k=0}^{i-1}\frac{d\phi_{i-1-k}(\eta)}{d\eta}\frac{d\phi_k(\eta)}{d\eta}\right]\right\}. \quad (3.4)$$

Thus,

$$\lim_{m\to+\infty}\sum_{i=1}^{m} G_i(\eta) = \hbar\sum_{i=1}^{+\infty}\left\{\frac{d^3\phi_{i-1}(\eta)}{d\eta^3} + \alpha\sum_{k=0}^{i-1}\phi_k(\eta)\frac{d^2\phi_{i-1-k}(\eta)}{d\eta^2} + \beta\left[(1-\chi_i) - \sum_{k=0}^{i-1}\frac{d\phi_{i-1-k}(\eta)}{d\eta}\frac{d\phi_k(\eta)}{d\eta}\right]\right\}$$

$$= \hbar\left\{\frac{d^3}{d\eta^3}\sum_{i=1}^{+\infty}\phi_{i-1}(\eta) + \alpha\sum_{i=1}^{+\infty}\sum_{k=0}^{i-1}\phi_k(\eta)\frac{d^2\phi_{i-1-k}(\eta)}{d\eta^2}\right.$$

$$+\beta\left[\sum_{i=1}^{+\infty}(1-\chi_i)-\sum_{i=1}^{+\infty}\sum_{k=0}^{i-1}\frac{\mathrm{d}\phi_{i-1-k}(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta}\right]\Bigg\}$$

$$=\hbar\left\{\frac{\mathrm{d}^3}{\mathrm{d}\eta^3}\sum_{i=0}^{+\infty}\phi_i(\eta)+\alpha\sum_{k=0}^{+\infty}\sum_{j=k+1}^{+\infty}\phi_k(\eta)\frac{\mathrm{d}^2\phi_{j-1-k}(\eta)}{\mathrm{d}\eta^2}\right.$$

$$\left.+\beta\left[1-\sum_{k=0}^{+\infty}\sum_{j=k+1}^{+\infty}\frac{\mathrm{d}\phi_{j-1-k}(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta}\right]\right\}$$

$$=\hbar\left\{\frac{\mathrm{d}^3}{\mathrm{d}\eta^3}\left[\sum_{k=0}^{+\infty}\phi_k(\eta)\right]+\alpha\sum_{k=0}^{+\infty}\phi_k(\eta)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left[\sum_{k=0}^{+\infty}\phi_k(\eta)\right]\right.$$

$$\left.+\beta\left[1-\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\sum_{k=0}^{+\infty}\phi_k(\eta)\right)^2\right]\right\}. \tag{3.5}$$

Since $\hbar\neq 0$, we have by (3.3) and (3.5) that

$$\frac{\mathrm{d}^3}{\mathrm{d}\eta^3}\left[\sum_{k=0}^{+\infty}\phi_k(\eta)\right]+\alpha\sum_{k=0}^{+\infty}\phi_k(\eta)\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left[\sum_{k=0}^{+\infty}\phi_k(\eta)\right]+\beta\left[1-\left(\frac{\mathrm{d}}{\mathrm{d}\eta}\sum_{k=0}^{+\infty}\phi_k(\eta)\right)^2\right]=0. \tag{3.6}$$

Furthermore, from (2.3) and (2.11), we have

$$\sum_{k=0}^{+\infty}\phi_k(0)=0,\quad \sum_{k=0}^{+\infty}\phi_k'(0)=0,\quad \sum_{k=0}^{+\infty}\phi_k'(+\infty)=1. \tag{3.7}$$

Therefore owing to (3.6) and (3.7), if the infinite series (2.42) converges, it must be a solution of equations (1.1) and (1.2). Thus, we only need to concentrate on selecting the appropriate initial guess approximation $f_0(\eta)$, auxiliary linear operator $\mathscr{L}$ and auxiliary parameter $\hbar$ to ensure the infinite series (2.42) is convergent.

Notice that the infinite sequence (2.42) gives a family of explicit analytic solutions in three parameters $\varpi,\gamma\,(\gamma>0)$ and $\hbar\,(\hbar\neq 0)$. Certainly, some among them may converge to $f(\eta)$ but others might not, dependent upon their values. Moreover, some of them might be 'better' than others; we emphasize that it is the homotopy analysis method which provides us with great freedom and large flexibility to select 'better' values of ϖ,γ and $\hbar$ to ensure the related series (2.42) convergent to $f(\eta)$, although this kind of freedom has its negative side which we will discuss later in this paper.

The essence of any an approximation approach is to express a solution of a problem as a complete set of base functions. From (2.42), our current base functions are

$$1,\eta,\eta^m\exp(-n\gamma\eta)\quad(n\geqslant 1, m\geqslant 0). \tag{3.8}$$

According to (1.7), if $f'(\eta)\to 1$ exponentially as $\eta\to+\infty$, we have

$$1-f'(\eta)\propto\eta^{-(2\beta+1)}\exp(-\tfrac{1}{2}\eta^2). \tag{3.9}$$

Notice that we can rewrite the term $\exp(-\eta^2/2)$ in the following form:

$$\exp(-\tfrac{1}{2}\eta^2)=\exp(-\gamma\eta)\exp(-\tfrac{1}{2}\eta^2+\gamma\eta)$$

$$=\exp(-\gamma\eta)\left[1+(-\tfrac{1}{2}\eta^2+\gamma\eta)+\frac{1}{2!}\left(-\tfrac{1}{2}\eta^2+\gamma\eta\right)^2+\frac{1}{3!}\left(-\tfrac{1}{2}\eta^2+\gamma\eta\right)^3+\cdots\right].$$

Therefore, $f'(\eta)$ can be expressed by the set of base functions (3.8) if $f'(\eta)\to 1$, indeed exponentially as $\eta \to +\infty$. However, if $f'(\eta) \to 1$ algebraically as $\eta \to +\infty$, obviously, $f'(\eta)$ cannot be expressed by the set of base functions (3.8). Therefore, (3.8) is complete only for the solutions possessing the property $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$. In this way, we can exactly find if a solution of equations (2.1) and (2.2) indeed possesses the property $f'(\eta) \to 1$ exponentially as $\eta \to +\infty$.

Secondly, if the sequence (2.42) converges, its second-order derivatives with respect to η at $\eta = 0$, say,

$$\sum_{k=0}^{+\infty} \phi_k''(0), \tag{3.10}$$

must be convergent, too. From (A 11), we have its corresponding mth-order approximation

$$\sigma_m = \sum_{k=0}^{m} \phi_k''(0) = \sum_{k=0}^{m} \sum_{n=1}^{k+1} c_{k,n}^0. \tag{3.11}$$

The first- and second-order approximations are

$$\sigma_1 = (\gamma+\varpi)(1+\hbar) - \frac{\hbar}{\gamma}\left(\tfrac{1}{2}\alpha + \tfrac{3}{2}\beta\right) + \frac{\varpi\hbar}{\gamma^2}\left(\tfrac{1}{2}\alpha + \tfrac{3}{2}\beta\right) + \frac{\varpi^2\hbar}{\gamma^3}\left(\frac{\alpha}{4} + \frac{\beta}{4}\right),$$

$$\begin{aligned}\sigma_2 = {} & (\gamma+\varpi)(1+\hbar)^2 - \frac{\hbar}{\gamma}(\alpha+3\beta) + \frac{\varpi\hbar}{\gamma^2}(\alpha+3\beta) + \frac{\varpi\hbar^2}{\gamma^2}\left(\tfrac{3}{4}\alpha + \tfrac{7}{4}\beta\right) \\ & + \frac{\varpi^2\hbar}{\gamma^3}\left(\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta\right) + \frac{\varpi^2\hbar^2}{\gamma^3}\left(\tfrac{3}{8}\alpha + \tfrac{3}{8}\beta\right) - \frac{\hbar^2}{\gamma^3}\left(\tfrac{5}{6}\alpha^2 + 3\alpha\beta + \tfrac{8}{3}\beta^2\right) \\ & + \frac{\varpi\hbar^2}{\gamma^4}\left(\tfrac{35}{24}\alpha^2 + \tfrac{43}{8}\alpha\beta + \tfrac{59}{12}\beta^2\right) + \frac{\varpi^2\hbar^2}{\gamma^5}\left(\tfrac{101}{72}\alpha^2 + \tfrac{95}{24}\alpha\beta + \tfrac{37}{18}\beta^2\right) \\ & + \frac{\varpi^3\hbar^2}{\gamma^6}\left(\tfrac{35}{108}\alpha^2 + \tfrac{37}{72}\alpha\beta + \tfrac{41}{216}\beta^2\right),\end{aligned}$$

respectively. Our calculations indicate that σ_m contains the term $(\gamma+\varpi)(1+\hbar)^m$. Thus, $\hbar$ must belong to a subset of the region

$$|1+\hbar| \leqslant 1. \tag{3.12}$$

Notice that we define $\hbar \neq 0$ in (2.1) and (2.2). Therefore, we are sure that both of the infinite series (2.42) and (3.10) diverge when $\hbar \geqslant 0$ or $\hbar < -2$.

Let

$$f_m(\eta) = \sum_{k=0}^{m} \phi_k(\eta) \tag{3.13}$$

denote the mth-order of approximation of $f(\eta)$ and

$$E_m(\hbar, \gamma, \tilde{\omega}) = \int_0^{+\infty} [f_m'''(\eta) + \alpha f_m''(\eta) f_m(\eta) + \beta(1 - f_m'^2)]^2 \mathrm{d}\eta \tag{3.14}$$

its corresponding residual error. Obviously, if the series (2.42) converges, it must hold that

$$\lim_{m\to+\infty} E_m(\hbar, \gamma, \varpi) = 0. \tag{3.15}$$

Also, in order to ensure the series (2.42) converges sufficiently fast, it should hold that

$$E_m(\hbar, \gamma, \varpi) \leqslant E_{m-1}(\hbar, \gamma, \varpi). \tag{3.16}$$

Noticing that $E_m(\hbar, \gamma, \varpi)$ is dependent upon $\hbar, \gamma, \varpi$, we define the set

$$S_m = \{(\hbar, \gamma, \varpi) : E_m(\hbar, \gamma, \varpi) \leqslant E_{m-1}(\hbar, \gamma, \varpi)\}. \tag{3.17}$$

Then, if $\hbar, \gamma, \varpi$ belong to the set

$$S = \left\{ (\hbar, \gamma, \varpi) : (\hbar, \gamma, \tilde{\omega}) \in \bigcap_{m=1}^{+\infty} S_m, \quad \lim_{m \to +\infty} E_m(\hbar, \gamma, \varpi) = 0 \right\}, \tag{3.18}$$

the series (2.42) converges sufficiently fast to the solution of (1.1) and (1.2). Notice that such a set might have infinite elements, but for a definite solution only one among them is enough.

We consider here two different cases. The first is Blasius flow which corresponds to $\alpha = 1/2$, $\beta = 0$. The second is Falkner–Skan flow, corresponding to $\alpha = 1$. Our calculations indicate that, for Blasius flow, the series (2.42) is convergent at least in the region

$$-1 \leqslant \hbar < 0, \quad \gamma \geqslant 4, \quad \varpi = 0. \tag{3.19}$$

And for Falkner–Skan flow ($\alpha = 1$), the series (2.42) is convergent at least in the region

$$-1 \leqslant \hbar < 0, \quad \gamma \geqslant 5, \quad \varpi = 0,$$

but only of the parameter β in the restricted region $2 \geqslant \beta \geqslant \beta_0 = -0.1988$.

Also, all of our calculations indicate that the series (2.42) is convergent in the whole region $\eta \in [0, +\infty)$ to the solution $f(\eta)$ of (2.1) and (2.2), as long as the series (3.10) converges. Moreover, when the series (3.10) converges faster, the corresponding series (2.42) also converges faster.

4. The velocity profile

4.1. *Blasius flow*

As mentioned above, when $\alpha = 1/2$, $\beta = 0$, the series (2.42) is convergent at least in the region $-1 \leqslant \hbar < 0$, $\gamma \geqslant 4$ when $\varpi = 0$. For example, selecting $\varpi = 0$, $\hbar = -1$ and $\gamma = 4$, we get the 55th-order of approximation which agrees well with Howarth's (1938) numerical result, as shown in tables 1 and 2. Notice that, as shown in table 1, all of our analytic approximations at over 45th-order give $f''(0) = 0.332057$ which agrees well with Howarth's (1938) numerical value $f''(0) = 0.33206$. Here, we emphasize two points. First, different from Blasius' (1908), Bairstow's (1925) and Goldstein's (1930) series solutions, the series (2.42) converges now in the whole region $0 \leqslant \eta < +\infty$. Second, to our knowledge, it is the first time that a uniformly valid, purely analytic solution $f(\eta)$ and also an analytic value of $f''(0)$ has been given for the Blasius flow. Thus, even better than Liao's (1997*a*) power series, (2.42) can give a totally (or purely) analytic solution of the Blasius flow which is uniformly valid in the whole region $0 \leqslant \eta < +\infty$. This verifies well the validity and potential of the homotopy analysis method as an analytic tool for non–linear problems.

4.2. *Falkner–Skan flow*

When $\alpha = 1$, the series (2.42) describes the Falkner–Skan viscous flow. As proved by Hartree (1937) and Stewartson (1954), when $0 \leqslant \beta \leqslant 2$, the Falkner–Skan equation (1.6) has a unique solution whose first-order derivative tends to 1 exponentially. Setting $\varpi = 0$ for the sake of simplicity, we find that (2.42) is convergent at least in the region $-1 \leqslant \hbar < 0$, $\gamma \geqslant 5$ for any values $0 \leqslant \beta \leqslant 2$. For example, when

Order of approximations	$f''(0)$
5th	0.256390
10th	0.327756
15th	0.331256
20th	0.331851
25th	0.332004
30th	0.332040
35th	0.332052
40th	0.332055
45th	0.332057
50th	0.332057
55th	0.332057

TABLE 1. Analytic approximations of $f''(0)$ for Blasius flow ($\alpha = 1/2, \beta = 0$) when $\varpi = 0, \gamma = 4, \hbar = -1$.

η	10th-order	20th-order	30th-order	40th-order	50th-order	55th-order	Numerical results
0.4	0.128944	0.132650	0.132756	0.132763	0.132764	0.132764	0.1328
0.8	0.249855	0.264412	0.264288	0.264707	0.264709	0.264709	0.2647
1.2	0.369985	0.393075	0.393755	0.393772	0.393776	0.393776	0.3938
1.6	0.511611	0.514758	0.516680	0.516750	0.516756	0.516756	0.5168
2.0	0.667300	0.626372	0.629553	0.629754	0.629764	0.629764	0.6298
2.4	0.803701	0.727156	0.728494	0.728950	0.728980	0.728980	0.7290
2.8	0.898980	0.814839	0.810980	0.811429	0.811503	0.811503	0.8115
3.2	0.954007	0.885026	0.876124	0.875982	0.876066	0.876066	0.8761
3.6	0.981197	0.935172	0.924321	0.923315	0.923312	0.923312	0.9233
4.0	0.993004	0.966854	0.957245	0.955665	0.955518	0.955518	0.9555
4.4	0.997605	0.984622	0.977780	0.976154	0.975900	0.975900	0.9759
5.0	0.999581	0.995914	0.992920	0.991856	0.991599	0.991599	0.9916
6.0	0.999983	0.999708	0.999317	0.999092	0.999006	0.999006	0.9990
7.0	1.000000	0.999987	0.999961	0.999939	0.999926	0.999926	1.0000
8.0	1.000000	1.000000	0.999999	0.999998	0.999997	0.999997	1.0000

TABLE 2. Comparison of the analytic approximations of $f'(\eta)$ for Blasius flow ($\alpha = 1/2, \beta = 0$) given by $\varpi = 0$, $\gamma = 4$ and $\hbar = -1$ with Howarth's (1938) numerical results.

$\hbar = -1$, $\gamma = 5$, $\varpi = 0$, the series (2.42) converges sufficiently fast and the 20th-order approximation agrees very well with numerical results, as shown in figures 1, 2 and 3. In fact, when $1 \leqslant \beta \leqslant 2$, the 10th-order approximation can give quite accurate velocity profiles, as shown in figures 2 and 3. The corresponding values of $f''(0)$ of the 20th-order analytic approximation also agree well with Howarth's (1938) numerical results, as shown in table 3. Because we obtain in this paper a purely analytic solution, we can give the analytic formula for the value $f''(0)$ for different β in the region $0 \leqslant \beta \leqslant 2$

$$f''(0) = \kappa_0 + \sum_{m=1}^{20} \kappa_m \beta^m, \tag{4.1}$$

where the coefficients κ_m are given in table 4.

Using this analytic formula, it is easy for us to calculate $f''(0)$ for any values of

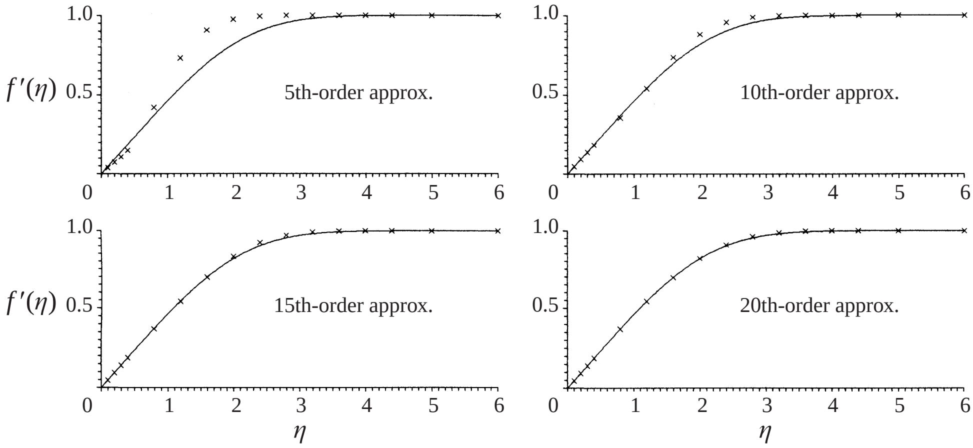

FIGURE 1. Comparisons of numerical results with analytic approximations upto 20th order ($\varpi = 0, \gamma = 5, \hbar = -1$) of the Falkner–Skan equation when $\beta = 0$: symbol; analytic approximations; solid line, numerical solutions given by Runge–Kutta's method ($f''(0) = 0.469601$).

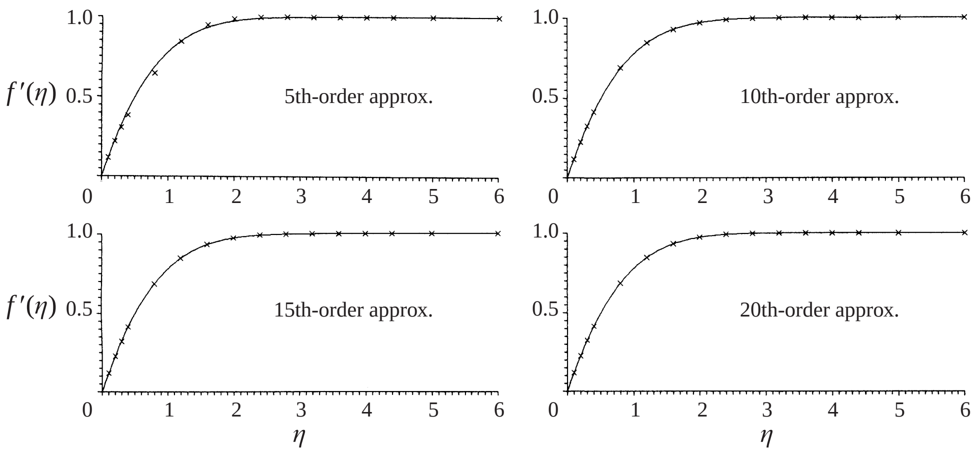

FIGURE 2. As figure 1 but for $\beta = 1$ ($f''(0) = 1.232588$).

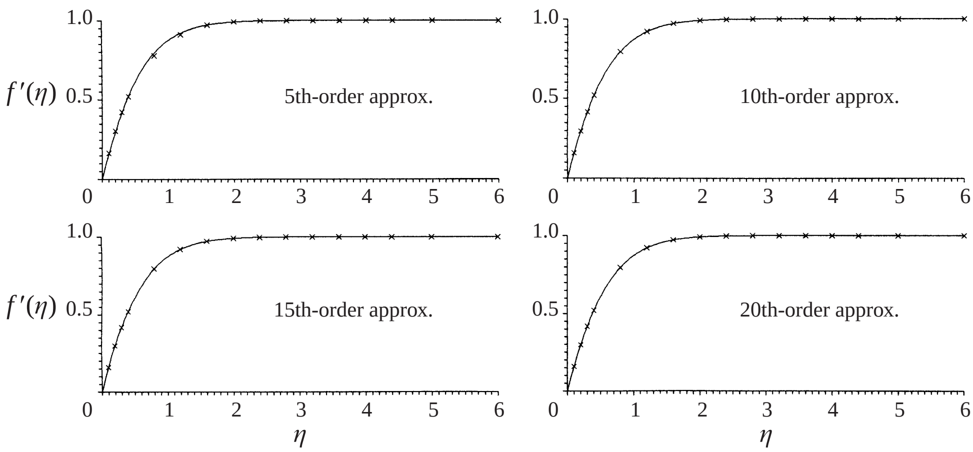

FIGURE 3. As figure 1 but for $\beta = 2$ ($f''(0) = 1.687218$).

β	10th-order	15th-order	20th-order	Numerical results
2.0	1.6865	1.6871	1.6872	1.6872
1.6	1.5171	1.5214	1.5215	1.5215
1.2	1.3315	1.3357	1.3358	1.3357
1.0	1.2308	1.2324	1.2327	1.2326
0.8	1.1221	1.1200	1.1203	1.1203
0.6	1.0011	0.9958	0.9957	0.9958
0.5	0.9338	0.9280	0.9276	0.9277
0.4	0.8604	0.8553	0.8544	0.8544
0.3	0.7792	0.7755	0.7748	0.7748
0.2	0.6883	0.6872	0.6869	0.6867
0.1	0.5852	0.5867	0.5871	0.5870
0.05	0.5282	0.5303	0.5310	0.5311
0.0	0.4669	0.4688	0.4695	0.4696

TABLE 3. Comparisons of analytic approximations $f''(0)$ for Falkner–Skan flow given by $\gamma = 5,\ \hbar = -1, \varpi = 0$ with numerical results (White 1991).

$\kappa_0 = 0.469471$,	$\kappa_{11} = 2.50815 \times 10^{-2}$,
$\kappa_1 = 1.29517$,	$\kappa_{12} = -4.97913 \times 10^{-3}$,
$\kappa_2 = -1.37974$,	$\kappa_{13} = 7.87925 \times 10^{-4}$,
$\kappa_3 = 2.19113$,	$\kappa_{14} = -9.87276 \times 10^{-5}$,
$\kappa_4 = -3.01070$,	$\kappa_{15} = 9.67527 \times 10^{-6}$,
$\kappa_5 = 3.21760$,	$\kappa_{16} = -7.26543 \times 10^{-7}$,
$\kappa_6 = -2.63773$,	$\kappa_{17} = 4.04235 \times 10^{-8}$,
$\kappa_7 = 1.67209$,	$\kappa_{18} = -1.57303 \times 10^{-9}$,
$\kappa_8 = -0.828893$,	$\kappa_{19} = 3.83118 \times 10^{-11}$,
$\kappa_9 = 0.324432$,	$\kappa_{20} = -4.40979 \times 10^{-13}$.
$\kappa_{10} = -0.100961$.	

TABLE 4. Coefficients of equation (4.1).

$0 \leqslant \beta \leqslant 2$. Thus, we do not need to interpolate the discrete values of $f''(0)$ given by Hartree (1937).

When $0 > \beta \geqslant \beta_0 = -0.19884$, the series (2.42) converges at least in the region

$$-1 \leqslant \hbar < 0, \quad \gamma \geqslant 5, \quad \varpi = 0 \tag{4.2}$$

but only to the family of solutions having the property $f''(0) \geqslant 0$, i.e. the family given by Hartree (1937). In general, when $0 > \beta \geqslant \beta_0 = -0.19884$, the series (2.42) converges slower than that when $0 \leqslant \beta \leqslant 2$, as shown in figures 4 and 5 ($\hbar = -1,\ \gamma = 5,\ \varpi = 0$), so that the 20th-order approximation formula (4.1) cannot give sufficiently accurate result for $f''(0)$ when $0 > \beta \geqslant \beta_0 = -0.19884$.

When $0 > \beta \geqslant \beta_0 = -0.19884$, Stewartson (1954) reported another family of solutions showing reversed flow. However, as mentioned above, when setting $\varpi = 0$, the series (2.42) converges only to Hartree's (1937) family of solutions for values of $\hbar$ and γ in the region $-1 \leqslant \hbar < 0, \gamma \leqslant 5$. Notice that, from (2.3), we have $f_0''(0) = \gamma + \varpi$. Thus, we can select negative values $\varpi < -\gamma$ to ensure that the initial guess approximation $f_0(\eta)$ has the property $f_0''(0) < 0$. We find that, in general, for a fixed value of γ, the larger the value of $|\varpi|$ ($\varpi < 0$) is, the smaller the value of $|\hbar|$ ($-2 < \hbar < 0$) should be so as to ensure the series (2.42) convergent. We attempted hundreds of combinations of different values of $\gamma, \varpi, \hbar$ and found, to some surprise,

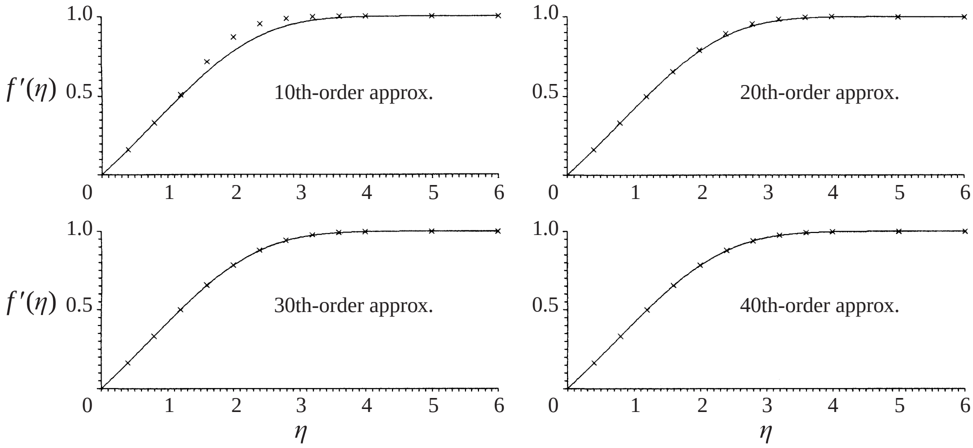

FIGURE 4. Comparisons of numerical result with analytic approximations of the Falkner–Skan equation when $\beta = -0.05$ at different orders given by $\varpi = 0, \gamma = 5, \hbar = -1$: symbol, analytic approximations; solid line, numerical solutions.

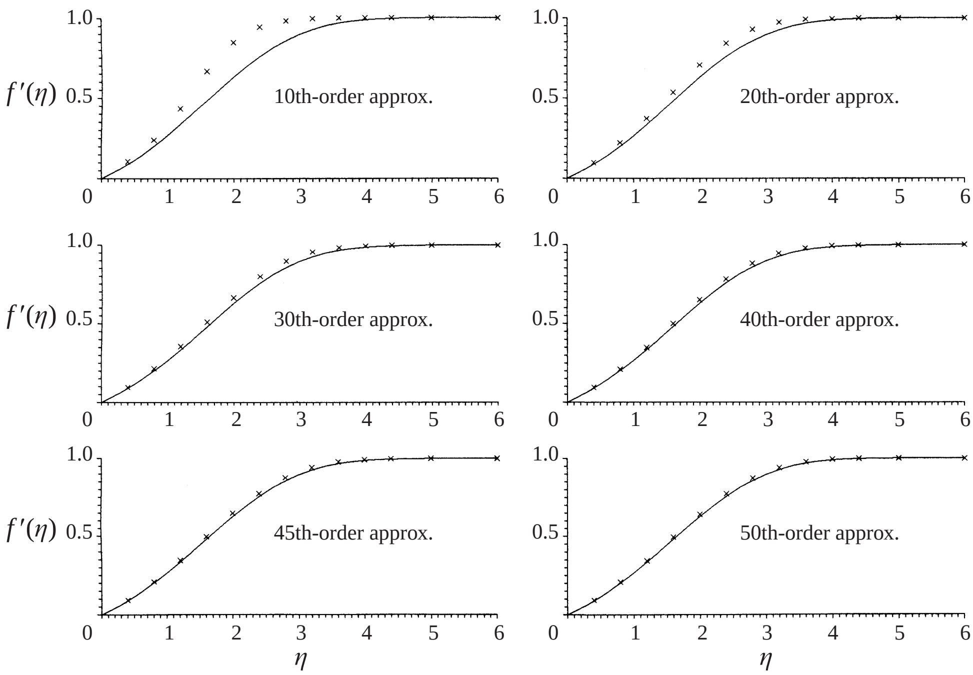

FIGURE 5. As figure 4 but for $\beta = -0.16$.

that, as long as the series (2.42) is convergent, it always converges to Hartree's (1937) family of solutions having the property $f''(0) \geqslant 0$. However, we cannot give a rigorous logical proof of it. In some cases such as $h = -1$, $\gamma = 4$, $\varpi = -5$, or $\hbar = -3/4$, $\gamma = 5$, $\varpi = -8$, the series (4.24) converges sufficiently fast to the Hartree's (1937) family of solutions. In other cases such as $\gamma = 3, \varpi = -12, \hbar = -1/10$ or $\gamma = 4, \varpi = -13, \hbar = -1/12$, the series (2.42) first even seems to approach to a kind of reversed flow but finally still converges, although rather slowly, to Hartree's (1937) family of solutions.

When $\beta < \beta_0 = -0.19884$, Stewartson (1954) proved that all solutions of the Falkner–Skan equation (1.6) have the property $f' > 1$ in some regions, showing velocity overshoot, and further Libby & Liu (1967) gave some families of this kind of solution. However, all of our calculations indicate that the series (2.42) would seem to be divergent when $\beta < \beta_0 = -0.19884$, although we cannot logically prove it. Therefore, the series (2.42) would seem to be valid only in the region $2 \geqslant \beta \geqslant \beta_0$ and to converge only to Hartree's (1937) family of solutions having the property $f''(0) \geqslant 0$.

If above result were indeed true, it might lead to two contrary conclusions. The first would be that the solutions, showing either reversed flow given by Stewartson (1954) or velocity overshoot given by Libby & Liu (1967), would not possess the property that $f' \to 1$ exponentially as $\eta \to +\infty$, because their asymptotic expressions for the term $1-f'(\eta)$ would contain an algebraic term with a rather small coefficient B relative to A in (1.7). In this case, no numerical methods could examine this small algebraic term. If so, these solutions certainly could not be expressed by the base functions (3.8) so that the series (2.42) naturally could not converge to them. Notice that, as pointed out by Stewartson (1954), when $0 > \beta > \beta_0$, the width of the region of the reversed flow is '$\propto (-\beta)^{-1/2}$ and so tends to infinity as the main-stream velocity tends to become constant' ($\beta = 0$); and also, 'as $\beta \to 0$- the fluid in the region of reversed flow comes to rest'. However, it would seem hard for the author to image a kind of laminar viscous flow over a semi-infinite flat plate which contains a layer of fluid at rest with infinite width but whose velocity exponentially increases to 1. Moreover, if this kind of solution were indeed to exist, there would exist two solutions to the Falker–Skan equation (1.6) in the case $\beta = 0$ (another form of Blasius equation). However, this would seem to be contrary to Hartree's (1937) and especially Weyl's (1942*a*, *b*) logic proof that there exists a unique solution when $\beta = 0$. The second conclusion is that the Stewartson's (1954) reversed flow solution and Libby & Liu's (1967) velocity overshoot solutions would have indeed the property that $f' \to 1$ exponentially as $\eta \to +\infty$ but our current approach would fail to give them. Notice that the homotopy analysis method provides us with rather great freedom to select the initial guess approximation $f_0(\eta)$, auxiliary linear operator $\mathscr{L}$ and the auxiliary parameter $\hbar$. We emphasize that it is just due to this kind of great freedom that the homotopy analysis method can overcome the restrictions of the perturbation techniques. However, on the other side, this kind of freedom might become a disadvantage of the homotopy analysis method, especially when there exist multiple solutions. One of the reasons is that there might exist a lot (probably an infinite number) of appropriate initial guess approximations and auxiliary linear operators $\mathscr{L}$ for a given problem, and the fact that a group of initial guess approximations and auxiliary linear operators $\mathscr{L}$ fail to give a series convergent to a definite solution does not mean that other will also fail to do so. For example, even using the new kind of auxiliary linear operator

$$\mathscr{L} = \frac{\partial^3}{\partial \eta^3} + 2\gamma \frac{\partial^2}{\partial \eta^2} + \gamma^2 \frac{\partial}{\partial \eta} \tag{4.3}$$

and the same initial guess approximation as (2.3), we still fail to find a series convergent for $\beta < \beta_0 = -0.19884$. However, we cannot logically prove that all possible initial guess approximations and auxiliary linear operators $\mathscr{L}$ fail to do so. What we are sure of is that Hartree's (1937) family of solutions showing neither reversed flow nor velocity overshoot has the property that $f' \to 1$ exponentially as $\eta \to +\infty$. We are also sure that using the initial guess approximation (2.3) and the auxiliary linear operator (1.12) we can apply the homotopy analysis method to give a uniformly valid, totally analytic solution of the Falkner–Skan equation (1.6) in the region $2 \geqslant \beta \geqslant \beta_0 = -0.19884$, which has the property $f''(0) \geqslant 0$. However, we are still not sure whether the solution showing reversed flow or velocity overshoot indeed possesses the property $f' \to 1$ exponentially as $\eta \to +\infty$, unless we can find an analytic series convergent to it by means of the homotopy analysis method or some other analytic techniques.

5. Conclusions and discussion

In this paper, we apply the homotopy analysis method (HAM) to obtain an explicit, totally analytic, uniformly valid solution (2.42) of a class of two-dimensional laminar viscous flow over a semi-infinite flat plate governed by the Falkner–Skan equation (1.6) or Blasius equation (1.3) under the boundary condition (1.2). We prove that, as long as the series (2.42) is convergent, it must converge to one of (1.1) and (1.2). Also, different from Blasius' (1908), Bairstow's (1925) and Goldstein's (1930) series solution for Blasius flow, this analytic solution is valid in the whole region $0 \leqslant \eta \leqslant +\infty$. We show that, for the Blasius flow ($\alpha = 1/2,\ \beta = 0$), the series (2.42) is convergent at least when $\varpi = 0, -1 \leqslant \hbar < 0, \gamma \geqslant 4$, and the 55th-order approximation given by $\varpi = 0, \hbar = -1, \gamma = 4$ agrees very well with Howarth's (1938) numerical results. For Falkner–Skan flow ($\alpha = 1$), the series (2.42) is convergent when $2 \geqslant \beta \geqslant \beta_0 = -0.19884$ at least in the region $\varpi = 0, -1 \leqslant \hbar < 0, \gamma \geqslant 5$, but only to the Hartree's (1937) family of solutions having the property $f''(0) \geqslant 0$. Although the series (2.42) is invalid for $\beta < \beta_0$; and also, cannot give families of solutions showing either reversed flow or velocity overshoot, to our knowledge thus it is the first time that such an explicit, uniformly valid, totally analytic solution to Falkner–Skan equation (1.6) when $2 \geqslant \beta \geqslant \beta_0 = -0.19884$ has been given. This verifies the validity and the potential of the homotopy analysis method as a new kind of analytic tool for nonlinear problems in fluid mechanics.

Notice that, using only the first four known coefficients

$$b_{0,0}^0 = \frac{\varpi}{\gamma^2} - \frac{1}{\gamma}, \quad b_{0,0}^1 = 1, \quad b_{0,1}^0 = -\frac{\varpi}{\gamma^2} + \frac{1}{\gamma}, \quad b_{0,1}^1 = -\frac{\varpi}{\gamma},$$

we can apply the recurrence formulae (2.20)–(2.27) and some other related expressions to calculate all coefficients $b_{m,n}^k$ of the solution (2.42). Although these recurrence formulae appear more complex than the recurrence formula (1.5) given by Blasius, it is now easy for us to calculate the coefficients $b_{m,n}^k$ by computers and symbolic calculation software such as *MATHEMATICA, MAPLE* and so on. Notice that Blasius flow is a special case of the equations (1.1) and (1.2). However, our analytic solution (2.42) is different from Blasius' power series solution (1.4) in two points. First, Blasius' power series solution (1.4) is a semi-numerical and semi-analytic ones because $\sigma = f''(0)$ in the formula (1.4) must be given by numerical approaches, but our solution (2.42) is a totally analytic solution and no coefficients $b_{m,n}^k$ need be given by numerical techniques. Secondly, Blasius' power series solution is valid in a restricted region but (2.42) is valid in the whole region $\eta \in [0, +\infty)$.

Compared with perturbation techniques, the homotopy analysis method has the following advantages. First, it does not depend on small parameters. Secondly, it provides us with great freedom and flexibility to select appropriate initial guess approximations, auxiliary linear operators and the auxiliary non-zero parameter $\hbar$. This kind of freedom and flexibility not only implies great potential for us to further improve the homotopy analysis method itself, but also provides us with a greater possibility to ensure that the related infinite sequence of approximations converge, and also to select 'better' ones from the family of approximations in more general forms. We believe that larger freedom and being more general means better. Independent of small parameters, the homotopy analysis method might become a new analytic tool for nonlinear problems in science and engineering, although it needs further improvements.

When $\beta \geqslant 0$, the Falkner–Skan equation (1.6) has a unique solution. In this simple case, we only need select appropriate combinations of $\varpi, \hbar, \gamma$ to ensure the series (2.42) convergent, because we have proved in §3 that a convergent series of (2.42) must converge to one of its solutions. Our calculations indicate that at least when

$$-1 \leqslant \hbar < 0, \quad \gamma \geqslant 5, \quad \varpi = 0,$$

the series (2.42) indeed converges to the unique solution of the Falkner–Skan equation ($\beta \geqslant 0$). In case $\varpi = 0$, different values of γ corresponds to different initial guess approximations $f_0(\eta)$ and different 'auxiliary' linear operators $\mathscr{L}$, and moreover, different values of $\hbar$ give different 'deformations' governed by the zeroth-order deformation equations (2.1) and (2.2). This means that a lot of pairs of $\gamma(\gamma \geqslant 5)$ and $\hbar(-1 \leqslant \hbar < 0)$, or in other words, a lot of initial guess approximations $f_0(\eta)$ and auxiliary linear operators $\mathscr{L}$ and also many kind of related deformations, can make the series (2.42) convergent to the unique solution when $\beta \geqslant 0$. How should we understand this? What we should emphasize here is that the solution (2.42) is given in the form of a kind of limit and the values of ϖ, γ and $\hbar$ determine the approach of tending to the limit. In essence, this is similar to the limit of a real function having two variables

$$\Pi = \lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x}. \tag{5.1}$$

It is well-known that the result Π of the above limit is strongly dependent upon the way or the approach to how the point (x, y) tends to (0,0). Assume that the point (x, y) tends to (0,0) along a path defined by

$$y = x^{\epsilon}, \quad \epsilon > 0, \tag{5.2}$$

we have

$$\Pi = \begin{cases} 1, & \epsilon > 1 \\ 2, & \epsilon = 1 \\ +\infty, & 0 < \epsilon < 1. \end{cases} \tag{5.3}$$

Here, we emphasize two points. First, the limit (5.1) is strongly dependent upon the path leading to the unique point (0,0). Secondly, there exist an infinite number of paths corresponding to $\epsilon > 1$, which give the same result $\Pi = 1$. In other words, there exist an infinite number of paths along which we get the same result $\Pi = 1$. It means that in case of $\epsilon > 1$ the limit (5.1) is independent of the path $y = x^{\epsilon}$. Similarly, although at $p = 1$ the zeroth-order deformation equations (2.1) and (2.2) have a unique solution when $\beta \geqslant 0$, there exist, however, an infinite number of different approaches to tend to this unique solution. Some among these approaches make the series (4.24) convergent faster and therefore are better than others, but some of them (for example, those in the case $|1+\hbar| > 1$) are so bad that the corresponding sequence is divergent. And correspondingly, there should exist some better values of γ and $\hbar$. In other words, there should exist some better auxiliary linear operators, better initial guess approximations and better zeroth-order deformation equations which make the related approximation sequence converge fast enough to the unique solution. We emphasize that it is the proposed homotopy analysis method (HAM) which provides us with such a possibility and the freedom to select the operator (1.12), which is more general than (1.11) used by Liao (1997*a*), as our auxiliary linear operator, that we can get the uniformly valid, totally analytic solution (2.42).

However, when $\beta < 0$, the Falkner–Skan equation has multiple solutions. In this case, the series (2.42) would seem to converge only to Hartree's (1937) family of solutions having the property $f''(0) \geqslant 0$. This result might lead to two contrary conclusions. The first would be that Stewartson's (1954) family of solutions, showing reversed flow, and Libby & Liu's (1967) family of solutions, showing velocity overshoot, would not possess the property $f' \to 1$ exponentially as $\eta \to +\infty$, because such an asymptotic property cannot be rigorously examined by numerical techniques and the series (2.42) can express only the solution having such a property. The contrary conclusion would be that Stewartson's (1954) and Libby & Liu's (1967) families of solutions would indeed possess the property $f' \to 1$ exponentially as $\eta \to +\infty$, but by selecting the initial guess approximation (2.3) and the form of auxiliary linear operator (1.12), our current approach could not apply the homotopy analysis method to express them, because in theory there would exist an infinite number of combinations of initial guess approximations and auxiliary linear operators. If the second conclusion were right, we should find other suitable auxiliary linear operators and initial guess approximations to give the solutions showing reversed flow or velocity overshoot.

This also demonstrates the restrictions of the homotopy analysis method. In general, when a nonlinear equation has a unique solution, it seems easy to apply the homotopy analysis method, because it can be generally proved that as long as the approximation sequence given by the homotopy analysis method is convergent, it must converge to a solution of the equation under consideration. In this case, although in general there exist an infinite number of possible combinations of initial guess approximations, auxiliary linear operators and the auxiliary parameter $\hbar$, only one among them is enough for us. However, if there exist multiple solutions, it seems hard to determine which combinations of initial guess approximations and auxiliary linear operators might give a definite solutions. This situation would become more serious when the considered problem has infinite number of solutions. Thus, the homotopy analysis method would be incomplete, unless we could improve it and give a more definite way to select the initial guess approximations and auxiliary linear operators. This also shows the negative side of the freedom in selecting initial guess approximations and auxiliary linear operators, although it is just by this kind of freedom the homotopy analysis method can overcome the restrictions of perturbation techniques.

Although the proposed homotopy analysis method approach does not give solutions showing either reversed flow or velocity overshoot, it successfully gives, for the first time (to our knowledge), Hartree's (1937) family of uniformly valid, totally analytic solutions of the Falkner–Skan equation. Also, we obtain corresponding analytic formula (4.1) for $f''(0)$ for $\beta \geqslant 0$ so that no interpolation is needed. Furthermore, we are quite sure that when $\beta < 0$ at least Hartree's (1937) family of solutions indeed possesses the property that $f' \to 1$ exponentially as $\eta \to +\infty$. Notice that, as pointed out by White (1991), perturbation techniques have not been successful in giving a sufficiently accurate analytic approximation of the viscous flow past a sphere valid in a large Reynolds number region. Notice also that in essence the viscous flow over a semi-infinite plate governed by (1.1) and (1.2) has many of the same physical properties as the viscous flow past a sphere. Thus, the success of the homotopy analysis method in solving Blasius flow ($\alpha = 1/2, \beta = 0$) and Falkner–Skan flow

($\alpha = 1, \beta_0 \leqslant \beta \leqslant 2$) problems might suggest the possibility of applying it to some other unsolved problems in fluid mechanics, such as the viscous flow past a sphere (in this case there exists a unique solution for steady flow) and so on.

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Appendix. Derivation of coefficients appearing in (2.15)

(i) The initial approximation $\phi_0(\eta) = f_0(\eta)$ defined by (2.3) has the same structure as (2.15), where the real function $\Psi_{m,k}(\eta)$ is defined by (2.16)–(2.19).

(ii) If we assume that the first $(m-1)$ solutions $\phi_k(\eta)(k = 0, 1, 2, 3, \ldots, m-1)$ have the same structure as (2.15), then we can prove that $\phi_m(\eta)$ has the same structure as (2.15).

To prove this, we define for simplicity

$$\lambda_{i,j}^k = \begin{cases} 0, & i = j = 0, \quad k \geqslant 2, \\ 0, & i > 0, \quad j = 0, \quad k \geqslant 1, \\ 0, & j > i + 1, \\ 0, & k > 2(i+1) - j, \\ 1, & \text{otherwise.} \end{cases} \tag{A 1}$$

Then, $\Psi_{m,k}(\eta)$ can be simply rewritten as

$$\Psi_{m,k}(\eta) = \sum_{i=0}^{2(m+1)-k} \lambda_{m,k}^i b_{m,k}^i \eta^i, \quad 0 \leqslant k \leqslant m+1 \tag{A 2}$$

for both $k = 0$ and $k \neq 0$. Thus, we have for $0 \leqslant k \leqslant m+1$ that

$$\begin{aligned} \Psi'_{m,k}(\eta) &= \sum_{i=1}^{2(m+1)-k} i\lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-1} \\ &= \sum_{i=0}^{2m+1-k} (i+1)\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1}\, \eta^i = \sum_{i=0}^{2(m+1)-k} (i+1)\lambda_{m,k}^{i+1}\, b_{m,k}^{i+1}\, \eta^i, \end{aligned} \tag{A 3}$$

$$\begin{aligned} \Psi''_{m,k}(\eta) &= \sum_{i=2}^{2(m+1)-k} i(i-1)\, \lambda_{m,k}^i\, b_{m,k}^i\, \eta^{i-2} \\ &= \sum_{i=0}^{2m-k} (i+2)(i+1)\lambda_{m,k}^{i+2}\, b_{m,k}^{i+2}\, \eta^i = \sum_{i=0}^{2(m+1)-k} (i+2)(i+1)\lambda_{m,k}^{i+2}\, b_{m,k}^{i+2}\, \eta^i. \end{aligned} \tag{A 4}$$

According to (2.15), we have

$$\phi'_m(\eta) = \sum_{k=0}^{m+1} \left[\Psi'_{m,k} - k\gamma\Psi_{m,k}\right] \exp(-k\gamma\eta), \tag{A 5}$$

and

$$\phi''_m(\eta) = \sum_{k=1}^{m+1} \left[\Psi''_{m,k} - 2k\gamma\Psi'_{m,k} + (k\gamma)^2\Psi_{m,k}\right] \exp(-k\gamma\eta). \tag{A 6}$$

By (A 3) and (A 4), we get

$$\Psi'_{m,k} - (k\gamma)\Psi_{m,k} = \sum_{i=0}^{2(m+1)-k} a^i_{m,k}\eta^i, \quad 0 \leqslant k \leqslant m+1, \tag{A 7}$$

and

$$\Psi''_{m,k} - 2k\gamma\Psi'_{m,k} + (k\gamma)^2\Psi_{m,k} = \sum_{i=0}^{2(m+1)-k} c^i_{m,k}\,\eta^i, \quad 1 \leqslant k \leqslant m+1, \tag{A 8}$$

respectively, where

$$a^i_{m,k} = (i+1)\lambda^{i+1}_{m,k}\, b^{i+1}_{m,k} - (k\gamma)\lambda^i_{m,k}\, b^i_{m,k}, \tag{A 9}$$

and

$$c^i_{m,k} = (i+1)(i+2)b^{i+2}_{m,k}\,\lambda^{i+2}_{m,k} - 2(k\gamma)(i+1)\, b^{i+1}_{m,k}\,\lambda^{i+1}_{m,k} + (k\gamma)^2\, b^i_{m,k}\,\lambda^i_{m,k}. \tag{A 10}$$

Thus, by (A 6) and (A 8), it holds that

$$\phi''_m(\eta) = \sum_{k=1}^{m+1} \exp(-k\gamma\eta)\left(\sum_{i=0}^{2(m+1)-k} c^i_{m,k}\,\eta^i\right), \quad m \geqslant 1. \tag{A 11}$$

Differentiating the above expression with respect to η, we obtain

$$\phi'''_m(\eta) = \sum_{k=1}^{m+1} \exp(-k\gamma\eta)\left(\sum_{i=0}^{2(m+1)-k} d^i_{m,k}\eta^i\right), \quad m \geqslant 1, \tag{A 12}$$

where

$$d^i_{m,k} = (i+1)\lambda^{i+1}_{m,k}\, c^{i+1}_{m,k} - (k\gamma)\,\lambda^i_{m,k}\, c^i_{m,k}. \tag{A 13}$$

When $0 \leqslant k \leqslant m-1$, we get by (2.15), (A 2) and (A 11)

$$\begin{aligned}
\phi_{m-1-k}(\eta)\frac{\mathrm{d}^2\phi_k(\eta)}{\mathrm{d}\eta^2} &= \sum_{r=0}^{m-k}\exp(-r\gamma\eta)\left(\sum_{s=0}^{2(m-k)-r}\lambda^s_{m-1-k,r}\, b^s_{m-1-k,r}\,\eta^s\right) \\
&\quad\times\sum_{j=1}^{k+1}\exp(-j\gamma\eta)\left(\sum_{i=0}^{2(k+1)-j} c^i_{k,j}\eta^i\right) = \sum_{j=1}^{k+1}\sum_{r=0}^{m-k}\exp\left[-(j+r)\gamma\eta\right] \\
&\quad\times\left(\sum_{i=0}^{2(k+1)-j}\ \sum_{s=0}^{2(m-k)-r} c^i_{k,j}\, b^s_{m-1-k,r}\,\lambda^s_{m-1-k,r}\,\eta^{s+i}\right) = \sum_{n=1}^{m+1}\exp(-n\gamma\eta)\sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \\
&\quad\times\left(\sum_{i=0}^{2(k+1)-j}\ \sum_{s=0}^{2(m-k)-n+j} c^i_{k,j}\, b^s_{m-1-k,n-j}\,\lambda^s_{m-1-k,n-j}\,\eta^{s+i}\right) = \sum_{n=1}^{m+1}\exp(-n\gamma\eta)\sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \\
&\quad\times\left(\sum_{q=0}^{2(m+1)-n}\eta^q\sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} c^i_{k,j}\, b^{q-i}_{m-1-k,n-j}\,\lambda^{q-i}_{m-1-k,n-j}\right) = \sum_{n=1}^{m+1}\exp(-n\gamma\eta) \\
&\quad\times\sum_{q=0}^{2(m+1)-n}\eta^q\left(\sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}}\ \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} c^i_{k,j}\, b^{q-i}_{m-1-k,n-j}\,\lambda^{q-i}_{m-1-k,n-j}\right)
\end{aligned} \tag{A 14}$$

which further gives

$$\alpha \sum_{k=0}^{m-1} \phi_{m-1-k}(\eta) \frac{\mathrm{d}^2\phi_k(\eta)}{\mathrm{d}\eta^2} = \exp(-\gamma\eta) \sum_{q=0}^{2m+1} \delta_{m,1}^q \eta^q + \sum_{n=2}^{m+1} \exp(-n\gamma\eta) \left(\sum_{q=0}^{2(m+1)-n} \delta_{m,n}^q \eta^q \right), \tag{A 15}$$

where

$$\delta_{m,n}^q = \alpha \sum_{k=0}^{m-1} \sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} c_{k,j}^i \, b_{m-1-k,n-j}^{q-i} \, \lambda_{m-1-k,n-j}^{q-i}, \tag{A 16}$$

for $1 \leqslant n \leqslant m+1, 0 \leqslant q \leqslant 2(m+1)-n$. By a straightforward calculation, we obtain $\delta_{m,1}^{2m+1} = 0$ for $m \geqslant 1$. Thus, (A 15) becomes

$$\alpha \sum_{k=0}^{m-1} \phi_{m-1-k}(\eta) \frac{\mathrm{d}^2\phi_k(\eta)}{\mathrm{d}\eta^2} = \exp(-\gamma\eta) \sum_{q=0}^{2m} \delta_{m,1}^q \eta^q + \sum_{n=2}^{m+1} \exp(-n\gamma\eta) \left(\sum_{q=0}^{2(m+1)-n} \delta_{m,n}^q \eta^q \right). \tag{A 17}$$

Similarly, we have by (A 5) and (A 7)

$$\begin{aligned}
\frac{\mathrm{d}\phi_{m-1-k}(\eta)}{\mathrm{d}\eta} \frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta}
&= \sum_{r=0}^{m-k} \exp(-r\gamma\eta) \left(\sum_{s=0}^{2(m-k)-r} a_{m-1-k,r}^s \eta^s \right) \sum_{j=0}^{k+1} \exp(-j\gamma\eta) \left(\sum_{i=0}^{2(k+1)-j} a_{k,j}^i \, \eta^i \right) \\
&= \sum_{j=0}^{k+1} \sum_{r=0}^{m-k} \exp[-(j+r)\gamma\eta] \left(\sum_{i=0}^{2(k+1)-j} \sum_{s=0}^{2(m-k)-r} a_{k,j}^i \, a_{m-1-k,r}^s \, \eta^{s+i} \right) \\
&= \sum_{n=0}^{m+1} \exp(-n\gamma\eta) \sum_{j=\max\{0,n+k-m\}}^{\min\{n,k+1\}} \left(\sum_{i=0}^{2(k+1)-j} \sum_{s=0}^{2(m-k)-n+j} a_{k,j}^i a_{m-1-k,n-j}^s \eta^{s+i} \right) \\
&= \sum_{n=0}^{m+1} \exp(-n\gamma\eta) \sum_{j=\max\{0,n+k-m\}}^{\min\{n,k+1\}} \left(\sum_{q=0}^{2(m+1)-n} \eta^q \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} a_{k,j}^i \, a_{m-1-k,n-j}^{q-i} \right) \\
&= \sum_{n=0}^{m+1} \exp(-n\gamma\eta) \sum_{q=0}^{2(m+1)-n} \eta^q \left(\sum_{j=\max\{0,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} a_{k,j}^i a_{m-1-k,n-j}^{q-i} \right).
\end{aligned} \tag{A 18}$$

From (A 18) we get

$$\begin{aligned}
\beta(1-\chi_m) - \beta \sum_{k=0}^{m-1} \frac{\mathrm{d}\phi_{m-1-k}(\eta)}{\mathrm{d}\eta} \frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta} &= \beta(1-\chi_m) + \sum_{n=0}^{m+1} \exp(-n\gamma\eta) \sum_{q=0}^{2(m+1)-n} \Delta_{m,n}^q \eta^q \\
&= \left[\beta(1-\chi_m) + \sum_{q=0}^{2(m+1)} \Delta_{m,0}^q \, \eta^q \right] + \sum_{n=1}^{m+1} \exp(-n\gamma\eta) \sum_{q=0}^{2(m+1)-n} \Delta_{m,n}^q \eta^q,
\end{aligned} \tag{A 19}$$

where

$$\Delta_{m,n}^{q} = -\beta \sum_{k=0}^{m-1} \sum_{j=\max\{0,n+k-m\}}^{\min\{n,k+1\}} \sum_{i=\max\{0,q-2(m-k)+n-j\}}^{\min\{q,2(k+1)-j\}} a_{k,j}^{i}\, a_{m-1-k,n-j}^{q-i}. \tag{A 20}$$

From (A 1), (A 9), (2.12) and (A 20), straightforward calculations show that

$$\beta(1-\chi_m) + \sum_{q=0}^{2(m+1)} \Delta_{m,0}^{q}\, \eta^q = 0,\ \Delta_{m,1}^{2m+1} = 0, \tag{A 21}$$

so that we have

$$\begin{aligned} &\beta\left[(1-\chi_m) - \sum_{k=0}^{m-1} \frac{\mathrm{d}\phi_{m-1-k}(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta}\right] \\ &\quad = \exp(-\gamma\eta)\sum_{q=0}^{2m} \Delta_{m,1}^{q}\, \eta^q + \sum_{n=2}^{m+1} \exp(-n\gamma\eta) \sum_{q=0}^{2(m+1)-n} \Delta_{m,n}^{q}\, \eta^q. \end{aligned} \tag{A 22}$$

Owing to (2.13), (2.14) and (2.12), we have for $m \geqslant 1$

$$\begin{aligned} G_m(\eta) = \hbar\Bigg\{ &\frac{\mathrm{d}^3\phi_{m-1}(\eta)}{\mathrm{d}\eta^3} + \alpha\sum_{k=0}^{m-1}\phi_{m-1-k}(\eta)\frac{\mathrm{d}^2\phi_k(\eta)}{\mathrm{d}\eta^2} \\ &+ \beta\left[(1-\chi_m) - \sum_{k=0}^{m-1}\frac{\mathrm{d}\phi_{m-1-k}(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}\phi_k(\eta)}{\mathrm{d}\eta}\right]\Bigg\}. \end{aligned} \tag{A 23}$$

Substituting (A 15) and (A 22) into (A 23), we have

$$G_m(\eta) = \exp(-\gamma\eta)\sum_{q=0}^{2m} \varGamma_{m,1}^{q}\, \eta^q + \sum_{n=2}^{m+1}\exp(-n\gamma\eta)\left(\sum_{q=0}^{2(m+1)-n} \varGamma_{m,n}^{q}\eta^q\right), \tag{A 24}$$

where for $m \geqslant 1$,

$$\varGamma_{m,1}^{q} = \hbar(d_{m-1,1}^{q} + \delta_{m,1}^{q} + \Delta_{m,1}^{q}), \quad 0 \leqslant q \leqslant 2m-1, \tag{A 25}$$

$$\varGamma_{m,1}^{2m} = \hbar(\delta_{m,1}^{2m} + \Delta_{m,1}^{2m}), \tag{A 26}$$

$$\varGamma_{m,m+1}^{q} = \hbar(\delta_{m,m+1}^{q} + \Delta_{m,m+1}^{q}), \qquad 0 \leqslant q \leqslant m+1, \tag{A 27}$$

and for $2 \leqslant n \leqslant m$,

$$\varGamma_{m,n}^{q} = \begin{cases} \hbar(d_{m-1,n}^{q} + \delta_{m,n}^{q} + \Delta_{m,n}^{q}), & 0 \leqslant q \leqslant 2m-n \\ \hbar(\delta_{m,n}^{q} + \Delta_{m,n}^{q}), & 2m-n+1 \leqslant q \leqslant 2m-n+2 \\ 0, & \text{otherwise.} \end{cases} \tag{A 28}$$

Thus, substituting (A 24) into (2.10), we get the mth-order deformation equation

$$\mathscr{L}(\phi_m - \chi_m\phi_{m-1}) = \exp(-\gamma\eta)\sum_{q=0}^{2m}\varGamma_{m,1}^{q}\eta^q + \sum_{n=2}^{m+1}\exp(-n\gamma\eta)\left(\sum_{q=0}^{2(m+1)-n}\varGamma_{m,n}^{q}\eta^q\right). \tag{A 29}$$

In order to solve the above equation, we should first give solutions of the equation

$$Y'''(\eta) + \gamma Y''(\eta) = \eta^q \exp(-n\gamma\eta), \tag{A 30}$$

where $n \geqslant 1$ and $q \geqslant 0$ are integers. Here, we mention such a formula, say, for integers $q \geqslant 0$ and $n \geqslant 1$:

$$\int \eta^q \exp(-n\gamma\eta) \mathrm{d}\eta = -\exp(-n\gamma\eta) \sum_{j=0}^{q} \left(\frac{q!}{j!}\right) \frac{\eta^j}{(n\gamma)^{q-j+1}}. \tag{A 31}$$

We solve the differential equation (A 30) for two different cases, say, $n = 1$ and $n \geqslant 2$, respectively.

(*a*) When $n = 1$, (A 30) becomes

$$Y''(\eta) = \exp(-\gamma\eta) \int \exp(\gamma\eta)\eta^q \exp(-\gamma\eta)\mathrm{d}\eta = \frac{1}{q+1}\eta^{q+1} \exp(-\gamma\eta), \tag{A 32}$$

which further gives by (A 31) that

$$Y(\eta) = \exp(-\gamma\eta) \sum_{k=0}^{q+1} \mu_{1,k}^q \; \eta^k, \tag{A 33}$$

where

$$\mu_{1,k}^q = \frac{q!}{k!} \frac{(q-k+2)}{\gamma^{q-k+3}}, \quad 0 \leqslant k \leqslant q+1, \quad q \geqslant 0. \tag{A 34}$$

(*b*) When $n \geqslant 2$, (A 30) becomes by (A 31),

$$\begin{aligned} Y''(\eta) &= \exp(-\gamma\eta) \int \exp(\gamma\eta)\eta^q \exp(-n\gamma\eta)\mathrm{d}\eta \\ &= -\exp(-n\gamma\eta) \sum_{j=0}^{q} \frac{q!}{j!} \frac{\eta^j}{[(n-1)\gamma]^{q-j+1}} \end{aligned} \tag{A 35}$$

which further gives by (A 31) that

$$Y(\eta) = -\exp(-n\gamma\eta) \sum_{k=0}^{q} \mu_{n,k}^q \eta^k, \tag{A 36}$$

where

$$\begin{aligned} \mu_{n,k}^q = {} & \frac{q!}{k!} \frac{1}{(n-1)^{q-k+1}\gamma^{q-k+3}} \\ & \times \left\{ 1 - \left(1 - \frac{1}{n}\right)^{q-k+1} \left[(q-k+2) - (q-k+1)\left(1 - \frac{1}{n}\right) \right] \right\} \end{aligned} \tag{A 37}$$

for $0 \leqslant k \leqslant q, n \geqslant 2, q \geqslant 0$.

Thus, by (A 33) and (A 36), we obtain the following general solution of (A 29):

$$\begin{aligned} (\phi_m - \chi_m \phi_{m-1}) = {} & \exp(-\gamma\eta) \left[\sum_{q=0}^{2m} \Gamma_{m,1}^q \mu_{1,0}^q + \sum_{k=1}^{2m+1} \eta^k \left(\sum_{q=k-1}^{2m} \Gamma_{m,1}^q \mu_{1,k}^q \right) \right] \\ & - \sum_{n=2}^{m+1} \exp(-n\gamma\eta) \left[\sum_{k=0}^{2(m+1)-n} \eta^k \left(\sum_{q=k}^{2(m+1)-n} \Gamma_{m,n}^q \mu_{n,k}^q \right) \right] \\ & + C_1^m \exp(-\gamma\eta) + C_2^m \eta + C_3^m, \end{aligned} \tag{A 38}$$

where C_1^m, C_2^m and C_3^m are integral constants. Using the boundary conditions (2.11), we have

$$C_1^m = \sum_{q=0}^{2m} \Gamma_{m,1}^q(\gamma^{-1}\mu_{1,1}^q - \mu_{1,0}^q) + \sum_{n=2}^{m+1}\left[n\Gamma_{m,n}^0\mu_{n,0}^0 + \sum_{q=1}^{2(m+1)-n}\Gamma_{m,n}^q(n\mu_{n,0}^q - \gamma^{-1}\mu_{n,1}^q)\right], \quad \text{(A 39)}$$

$$C_2^m = 0, \quad \text{(A 40)}$$

$$C_3^m = -C_1^m - \sum_{q=0}^{2m}\Gamma_{m,1}^q\mu_{1,0}^q + \sum_{n=2}^{m+1}\sum_{q=0}^{2(m+1)-n}\Gamma_{m,n}^q\mu_{n,0}^q = -\sum_{q=0}^{2m}\gamma^{-1}\Gamma_{m,1}^q\mu_{1,1}^q$$
$$-\sum_{n=2}^{m+1}\left[(n-1)\Gamma_{m,n}^0\mu_{n,0}^0 + \sum_{q=1}^{2(m+1)-n}\Gamma_{m,n}^q(n\mu_{n,0}^q - \mu_{n,0}^q - \gamma^{-1}\mu_{n,1}^q)\right]. \quad \text{(A 41)}$$

Therefore, $\phi_m(\eta)$ has the same structure as (2.15) and the related coefficients $b_{m,n}^k$ can be calculated by the following recurrence formulae:

$$b_{m,0}^0 = \chi_m b_{m-1,0}^0 - \gamma^{-1}\sum_{q=0}^{2m}\Gamma_{m,1}^q\mu_{1,1}^q$$
$$-\sum_{n=2}^{m+1}\left[(n-1)\Gamma_{m,n}^0\mu_{n,0}^0 + \sum_{q=1}^{2(m+1)-n}\Gamma_{m,n}^q(n\mu_{n,0}^q - \mu_{n,0}^q - \gamma^{-1}\mu_{n,1}^q)\right], \quad \text{(A 42)}$$

$$b_{m,0}^1 = 0, \quad \text{(A 43)}$$

$$b_{m,1}^0 = \chi_m b_{m-1,1}^0 + \gamma^{-1}\sum_{q=0}^{2m}\Gamma_{m,1}^q\,\mu_{1,1}^q + \sum_{n=2}^{m+1}\left[n\Gamma_{m,n}^0\mu_{n,0}^0\right.$$
$$\left. + \sum_{q=1}^{2(m+1)-n}\Gamma_{m,n}^q(n\mu_{n,0}^q - \gamma^{-1}\mu_{n,1}^q)\right], \quad \text{(A 44)}$$

$$b_{m,1}^k = \chi_m\, b_{m-1,1}^k + \sum_{q=k-1}^{2m}\Gamma_{m,1}^q\,\mu_{1,k}^q, \quad 1 \leqslant k \leqslant 2m-1, \quad \text{(A 45)}$$

$$b_{m,1}^k = \sum_{q=k-1}^{2m}\Gamma_{m,1}^q\,\mu_{1,k}^q, \quad 2m \leqslant k \leqslant 2m+1, \quad \text{(A 46)}$$

$$b_{m,n}^k = \chi_m b_{m-1,n}^k - \sum_{q=k}^{2(m+1)-n}\Gamma_{m,n}^q\,\mu_{n,k}^q, \quad 0 \leqslant k \leqslant 2m-n, \quad 2 \leqslant n \leqslant m, \quad \text{(A 47)}$$

$$b_{m,n}^k = -\sum_{q=k}^{2(m+1)-n}\Gamma_{m,n}^q\,\mu_{n,k}^q, \qquad 2m-n+1 \leqslant k \leqslant 2m-n+2, \quad 2 \leqslant n \leqslant m, \quad \text{(A 48)}$$

$$b_{m,m+1}^k = -\sum_{q=k}^{m+1}\Gamma_{m,m+1}^q\,\mu_{m+1,k}^q, \qquad 0 \leqslant k \leqslant m+1, \quad \text{(A 49)}$$

where $m \geqslant 1, 0 \leqslant n \leqslant m+1$ and $0 \leqslant k \leqslant 2(m+1)-n$.

(iii) In (i), we pointed out that the initial approximation $\phi_0(\eta) = f_0(\eta)$ has the same structure as (2.15). In (ii), we not only deduce the recurrence formulas (A 42)–(A 49) but also rigorously prove that, if the first $(m-1)$ solutions $\phi_k(\eta)(k = 0, 1, 2, 3, \ldots, m-1)$ have the structure (2.15), then the mth solution $\phi_m(\eta)(m \geqslant 1)$ must have the same structure as (2.15), too. Therefore, due to (i) and (ii), all $\phi_k(\eta)(k \geqslant 0)$ have the same sturcture as (2.15). Thus, using the first four coefficients

$$b^0_{0,0} = \frac{\varpi}{\gamma^2} - \frac{1}{\gamma}, \quad b^1_{0,0} = 1, \quad b^0_{0,1} = -\frac{\varpi}{\gamma^2} + \frac{1}{\gamma}, \quad b^1_{0,1} = -\frac{\varpi}{\gamma^2}, \tag{A 50}$$

which are determined by the initial guess approximation $f_0(\eta)$ defined by (2.3), we can calculate all coefficients $b^k_{m,n}$, where $m \geqslant 1, 0 \leqslant n \leqslant m+1$ and $0 \leqslant k \leqslant 2(m+1) - n$.

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